

Where are the trapped surfaces?

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Abstract. We discuss the boundary of the spacetime region through each point of which a trapped surface passes, first in some simple soluble examples, and then in the self-similar Vaidya solution. For the latter the boundary must lie strictly inside the event horizon. We present a class of closed trapped surfaces extending strictly outside the apparent horizon.

1. Introduction

Roger Penrose tells the story of how the importance of closed trapped surfaces occurred to him while crossing a street in London. He then promptly forgot about the matter, but—fortunately—recalled it later in the day [1]. One can make a case for trapped surfaces being the single most important ingredient in gravitational collapse. Their popular acclaim has been clouded by the fact that they remain hidden behind the event horizon, but numerical relativists do encounter them, while the very existence of the event horizon may be in some doubt from quantum gravity.

Can one *define* a black hole using its trapped surfaces [2]? An important notion here is the dynamical horizon, a spacelike hypersurface foliated by marginally trapped surfaces [3]. But not enough is known about the uniqueness of such objects [4]. Indeed little is actually known about where trapped surfaces occur even in the simple Vaidya solution, which describes the collapse of a spherically symmetric cloud of incoherent radiation (or “null dust”) [5]. This is the situation that we have tried to remedy, with partial success.

Any dynamical horizon lies inside the trapping boundary, defined as the boundary of an inextendible spacetime region for which each point lies on some trapped surface [2]. In very simple cases this boundary is easily found. In the Vaidya spacetime it must lie somewhere between the event horizon and a spherically symmetric dynamical horizon, but coincides with neither [6, 7]. Here we will present a family of closed trapped surfaces that extend into the no-man’s land between them. The surfaces we consider are genuinely trapped, in that both of their null expansions are non-positive, whereas much of the recent literature concerns outer trapped surfaces for which no condition is imposed on the inner expansion. There are good reasons for this [8], but there are reasons for our choice too [2, 3]. The reader must not confuse the two. Eardley conjectured that the boundary of the region through which closed outer trapped surfaces pass is the event horizon [9], and for the Vaidya case this was proven by Ben-Dov [10].

2. Simple examples and how to address them

A spacetime where the trapping boundary can be identified almost by inspection is the recollapsing $k = 1$ Friedmann model. It is foliated by round 3-spheres and has a moment of time

symmetry for which the trace K of the second fundamental form vanishes. Any equatorial sphere in such a 3-sphere is minimal within the 3-sphere, and trapped if $K < 0$. The null expansions vanish at the moment of time symmetry, so we expect this to be the trapping boundary. But how do we prove that a closed trapped surface cannot extend partly below it?

Let e_i^a project the cotangent space onto the cotangent space of some spacelike surface of codimension two. Let $\bar{\xi}^i$ be the projection of an arbitrary spacetime vector field ξ^a , and let $K_{ij}{}^a$ be the shape tensor of the surface. Then

$$e_i^a e_j^b \nabla_a \xi_b = \bar{\nabla}_i \bar{\xi}_j + K_{ij}{}^a \xi_a . \quad (1)$$

The derivative on the right hand side uses the Levi-Civita connection of the first fundamental form. If we symmetrise and then contract with the first fundamental form we obtain

$$\frac{1}{2} \gamma^{ij} e_i^a e_j^b \mathcal{L}_\xi g_{ab} = \bar{\nabla}_i \bar{\xi}^i + H^a \xi_a , \quad (2)$$

where $H^a = \gamma^{ij} K_{ij}{}^a$ is the mean curvature vector. This is future directed and timelike for a trapped surface. Suppose that the vector field ξ^a is future directed and timelike too, so that $H^a \xi_a < 0$, and suppose that ξ^a is such that the left hand side of this equation is non-negative. If we integrate over a closed surface the divergence goes away and we have a contradiction. Either the surface extends into a region where ξ^a is spacelike, or else it is not trapped [11].

If the vector field ξ^a is hypersurface forming we can say more. Assume that there exist functions $F > 0$ and τ such that

$$\xi_a = -F \nabla_a \tau . \quad (3)$$

On a closed surface the function τ will assume a minimum. At that minimum

$$\bar{\nabla}_i \bar{\xi}^i = -\bar{\nabla}_i (F \bar{\nabla}^i \tau) = -F \bar{\nabla}_i \bar{\nabla}^i \tau - \bar{\nabla}_i F \bar{\nabla}^i \tau = -F \bar{\nabla}_i \bar{\nabla}^i \tau < 0 . \quad (4)$$

If ξ^a is such that the left hand side of eq. (2) is positive semidefinite we conclude that

$$H^a \xi_a > 0 \quad (5)$$

at the minimum of τ . Since ξ^a is timelike, the mean curvature vector H^a cannot be, and the surface cannot be trapped at the minimum of τ .

To return to the Friedmann model, it admits a future directed conformal Killing vector field for which the left hand side of eq. (2) is positive prior to the moment of time symmetry, and this is what we need in order to prove that the moment of time symmetry is indeed the trapping boundary. Through every point of this trapping boundary there passes a marginally trapped and indeed minimal closed surface. Examples where this is not so include open recollapsing Friedmann models where the trapping boundary is filled by minimal surfaces that are not closed. An intermediate case is provided by the BTZ wormhole, where minimal closed surfaces are dense in the trapping boundary, but do not pass through every point there [12].

The minimum argument can be used to give a stronger conclusion about the trapping boundary. At a minimum in τ

$$\gamma^{ij} e_i^a e_j^b \nabla_a \xi_b = -F \bar{\nabla}_i \bar{\nabla}^i \tau + H^a \xi_a . \quad (6)$$

The left hand side is a projection of the second fundamental form of a hypersurface orthogonal to ξ^a . The Laplacian is positive at the minimum, and the scalar product is negative if the surface is trapped, so the right hand side is negative. Therefore the second fundamental form must have at least one negative eigenvalue if a trapped surface otherwise to its future touches a hypersurface in a point. In 2+1 dimensional Minkowski space this is easy to see: A trapped “surface” is

then a spacelike curve bending downwards in inertial time, and a surface with positive definite second fundamental form is like a hyperboloid bending upwards in time. Obviously a trapped curve otherwise in its future cannot touch it in a point.

3. The self-similar Vaidya solution

We now turn to Vaidya's spacetime, with the metric

$$ds^2 = - \left(1 - \frac{2m(v)}{r} \right) dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 . \quad (7)$$

Einstein's equations read

$$G_{ab} = 8\pi T_{ab} = \frac{2\dot{m}}{r^2} \nabla_a v \nabla_b v . \quad (8)$$

The energy conditions are obeyed provided that $\dot{m} \geq 0$, where the dot denotes differentiation with respect to v . Otherwise the rate at which radiation comes in is at our disposal. We choose

$$m = \begin{cases} 0 & , \quad v \leq 0 \\ \mu v & , \quad 0 \leq v \leq M/\mu \\ M & , \quad v \geq M/\mu \end{cases} . \quad (9)$$

This describes a spherically symmetric shell of incoherent radiation entering flat spacetime from past null infinity, ending in a Schwarzschild black hole when the inflow stops—provided that $\mu > 1/16$, otherwise the result is a naked singularity [13, 14]. Other choices of mass function have been studied [15]. Our choice is special because the Vaidya region of the solution has a homothetic Killing vector

$$\eta = v\partial_v + r\partial_r , \quad (10)$$

whose flow lines are confined to hypersurfaces with constant x , where $x = v/r$. This extra symmetry will be quite helpful to us. We expect that the behaviour for a somewhat different mass function would be qualitatively similar but harder to get. The round spheres at $r = 2m$ are marginally trapped and foliate a regular dynamical horizon. We refer to it as the apparent horizon, because it is the apparent horizon in a spherically symmetric slicing of spacetime, and we do not wish to prejudge the issue whether there are other, non-spherically symmetric dynamical horizons present.

In a spherically symmetric spacetime the coordinate r has a meaning related to the area of the preferred round spheres. There is also a very special vector field known as the Kodama vector field ξ [16]. In our case it is $\xi = \partial_v$. It is not a Killing vector field in general, but it does define a direction in which the area of the preferred round spheres is constant. It is hypersurface forming, and future directed and timelike outside the apparent horizon. Through eq. (3) it defines a "Kodama time" τ in this region, and this can be used in the minimum argument presented in section 2. A trapped surface sticking out of the apparent horizon at some value of τ can only reach higher values in its exterior. Therefore trapped surfaces at values of τ smaller than the smallest value τ_Σ assumed on the apparent horizon itself are excluded. Setting $\tau = \tau_\Sigma$ defines a spacelike hypersurface Σ , touching the apparent horizon just where the latter joins the event horizon. It is spherically symmetric and defined by a function $v = v(r)$ obeying

$$\frac{dv}{dr} = \frac{1}{1 - \frac{2m}{r}} . \quad (11)$$

This is easily solved for in terms of elementary functions in the self-similar case [14]. In the flat region constant Kodama time means constant inertial time, and the hypersurface Σ will enter the flat region if and only if $\mu > 1/8$.

Is it possible that Σ is the trapping boundary we look for? In fact it cannot be, since the eigenvalues of its second fundamental form are $(k_1, k_2, k_3) = (k_1, 0, 0)$ where $k_1 > 0$. But we have already argued that such a hypersurface cannot be touched by marginally trapped surfaces. There must be a region to the future of Σ which is free of trapped surfaces, but at least we have constrained the trapping boundary from below. We have also proved that the latter must be spacelike close to the event horizon, since it is squeezed from below by the spacelike Σ .

4. Tongues sticking out of the apparent horizon

Is it possible that the trapping boundary coincides with the apparent horizon? In fact no [6]. An argument due to Galloway and Wald shows that marginally trapped round spheres in a spherically symmetric dynamical horizon can always be perturbed in such a way that they extend partly outside it, and such that the perturbation causes them to become trapped [17]. But their argument does not show how far below the apparent horizon these trapped surfaces extend. Here we will present a—non-optimal—construction which allows us to find closed trapped surfaces extending a finite distance away from the apparent horizon.

Our surfaces are defined as cross sections of a cone, by means of the two equations

$$v = \frac{k}{2\mu}r - v_0, \quad v = \frac{1}{2\mu}r + a(\theta). \quad (12)$$

When $a = 0$ this surface sits on the apparent horizon. When $|a|$ is large the surface may extend into the Schwarzschild or Minkowski regions of the solution, but we stick to the above definition—it defines a smoothly embedded surface in \mathbf{R}^4 , although its first fundamental form will not be smooth and its null expansions will jump at the boundaries between different regions.

The condition for outer trapping is the somewhat lengthy inequality

$$\begin{aligned} NH(k_+) = & -2\alpha r^3 + 2r^3 \left(1 - \frac{2m}{r}\right) + \frac{\alpha k - 2\mu}{k - 1} r^2 (a'' + a' \cot \theta) + \\ & + \frac{a'^2}{(k - 1)^2} \left(4\mu k m + 8\mu^2 r + \alpha k^2 m - 8\mu \alpha k r - k^2 \dot{m} r\right) + \\ & + \frac{a'^2}{(k - 1)^2} \left(1 - \frac{2m}{r}\right) \left(4\mu k r + \alpha k^2 r - k^2 m - r \left(1 - \frac{2m}{r}\right) k^2\right) + \\ & + \frac{k a'^2}{(k - 1)^2} \left(4\mu - k \left(1 - \frac{2m}{r}\right)\right) \frac{\alpha k - 2\mu}{k - 1} a' \cot \theta < 0, \end{aligned} \quad (13)$$

where $H(k_+)$ is the outer null expansion, N is a positive normalisation factor, and

$$\alpha = \frac{2\mu}{k} + \frac{(k - 1)^2 r^2}{k^2 a'^2} \left(1 - \sqrt{1 + \frac{4\mu k a'^2}{(k - 1)^2 r^2} - \left(1 - \frac{2m}{r}\right) \frac{k^2 a'^2}{(k - 1)^2 r^2}}\right). \quad (14)$$

To simplify the calculation we set

$$k = 1 + l(l + 1), \quad a(\theta) = a_0 + a_l P_l(\cos \theta), \quad (15)$$

where P_l is a Legendre polynomial. Together with eqs. (12) this defines a set of “tongues” that—as we will see—stick partly out of the apparent horizon. To first order in a the trapping condition becomes simply $a_0 > 0$, and the tongue extends partly outside the apparent horizon if there is a θ such that $a(\theta) < 0$. This is the perturbation considered by Galloway and Wald [17].

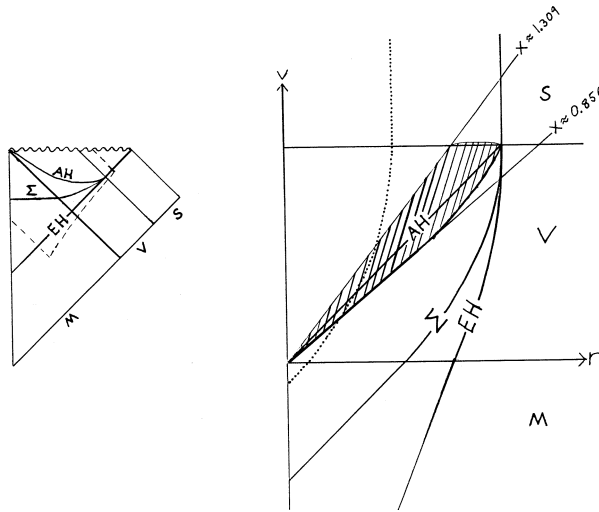


Figure 1. To the left is a Penrose diagram of a Vaidya spacetime. The region inside the dashed curve is shown as a $v - r$ -diagram to the right, and includes a part of the event horizon (EH), the spacelike part of the apparent horizon (AH), and the hypersurface Σ below which no trapped surfaces can extend. The tongues considered in the text are confined between two lines of constant x , and can extend only a little bit into the Schwarzschild region S. The trapped surfaces discussed earlier [6] are to the left of the dotted curve. The figure is for $\mu = 1/2$ only.

Our aim is now to adjust the parameters a_0 and a_l for every given v_0 in such a way that the extent to which the tongue sticks out of the horizon is maximised. To deal with the inequality (13) we had to fall back on Mathematica. General considerations about when the cone becomes tangential to hypersurfaces of constant Kodama time show that the possible extent of the tongue is larger the smaller l is, so we confined ourselves to $l = 1$ and $l = 2$. Then we investigated the two cases $\mu = 1/2$ and $\mu = 2$ in detail and found—for the case when the tongues remain in the Vaidya region of spacetime—that the most stringent condition from the inequality comes at the tips of the tongue, where $a' = 0$. But once this is so the inequality (13) is easily dealt with analytically. For $m = \mu v$, $l = 1$, the trapping conditions at the tips hold as long as

$$\mu = \frac{1}{2} : \quad 0.856 < x = \frac{v}{r} < 1.309 \quad (16)$$

$$\mu = 2 : \quad -0.417 < x = \frac{v}{r} < 0.75 . \quad (17)$$

This is then how far a maximally extended closed trapped tongue extends. The inner trapping condition holds throughout the allowed region. Thus the region occupied by trapped tongues is bounded by hypersurfaces of constant x . Given that the trapping condition becomes critical at the tips of the tongues this could have been predicted, because it is known that if the outer null expansion vanishes at a point it remains zero if the surface is moved by a homothety [18]. In the self-similar Vaidya solution this is generated by the vector field η , see eq. (10).

For the two values of μ that we investigated in detail we also found that the intrinsic curvature R of the tongue obeys $R > 0$ throughout the region where it is trapped, thus disproving a conjecture by Hayward [19].

So far we have avoided the complication that the tongues may extend into the Minkowski or Schwarzschild regions, where the mass function is different. This must now be dealt with. It is no longer true that the trapping condition is always at its most stringent at the tips of the tongue.

After a lengthy Mathematica calculation we found, however, that the upper Schwarzschild tip is usually the place where trapping fails first. The result is that the region into which the tongues can extend shrinks—as indeed it must, since its boundary must reach the point where the dynamical horizon meets the event horizon. See Fig. 1 for a summary of the calculation in the $\mu = 1/2$ case. For $\mu = 2$ the tongues can extend into the flat region.

It is not the case though that the Schwarzschild region makes life more difficult for all kinds of closed trapped surfaces. Indeed the flat and Vaidya region has locally trapped surfaces in the equatorial plane $\theta = \pi/2$. Topologically they are open disks meeting the Schwarzschild boundary in a circle. If they are carefully adjusted, and do not extend to large values of r , it is possible to close them in the Schwarzschild region while keeping them trapped, so the Schwarzschild region in a sense creates closed trapped surfaces that could not exist without it [6].

5. So where is the trapping boundary?

A large part of the no-man’s land between the dynamical horizon and the event horizon has now been occupied by the trapped region, but we have not been able to pin down its boundary precisely. Some things are clear: in a spherically symmetric spacetime the trapping boundary must itself be spherically symmetric, it cannot contain any marginally trapped surfaces, it cannot have a positive definite second fundamental form, and it must be spacelike close to the event horizon because it is squeezed from below by the spacelike hypersurface Σ . From our picture for $\mu = 1/2$ it is tempting to conjecture that a part of the boundary sits at $x = \text{constant}$. This part would be determined entirely by the behaviour of trapped surfaces inside the self-similar Vaidya region. But for $\mu = 2$ the trapping boundary is determined largely by what happens inside the Schwarzschild region—not by the local physics in the Vaidya region.

Our calculations have not addressed the question to what extent the dynamical horizon is unique. This also remains as an interesting question.

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